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AVERAGING OF AN ORTHOTROPIC ELASTIC PLATE WEAKENED BY PERIODIC HINGES OF FINITE STIFFNESS

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A formal asymptote of a solution of the title problem is constructed using the averaging method of N. S. Bakhvalov. The averaged equation is of an elliptic type; for small stiffness of the hinge, it is singularly perturbed, and for zero stiffness of the hinge, it is of a composite type. For the first boundary-value problem, a solution of the original problem is proved to converge to that of the limiting problem. A situation where natural boundary conditions are specified for the composite equation is treated. It is shown that the solution space of the homogeneous problem is infinite-dimensional.

Introduction. A formal asymptote of a solution of the title problem is obtained using the averaging method proposed by Bakhvalov [1]. Many problems have been solved by this method under the assumption that the contact between the component materials is ideal; studies in which the contact was not ideal are much fewer. Lene and Leguillon [2] studied the averaging problem of the theory of elasticity for a composite material in the case where viscous friction occurs in an elementary cell at the interface between the fiber and matrix. Assuming zero stiffness of the periodic hinges, Andrianov et al. [3] derived an averaged equation of a composite type for a particular case. It should be noted that the conjugation conditions in [3] were incorrect, namely, the requirement of zero jump in the transverse shear force was not formulated. Below, it is shown that averaging of a plate weakened by hinges of finite stiffness leads to an elliptic equation; for small stiffness of the hinge, this equation is singularly perturbed, and for zero stiffness of the hinges, it is of a composite type. Similar equations were treated in previous papers of the author (see, for instance, [4]). For the first boundary-value problem, a solution of the original problem is proved to converge to that of the limiting problem. A situation where the natural boundary conditions are specified for an equation of the composite type has been studied. It is shown that, in contrast to the elliptic case, the solution space of the homogeneous problem is infinite-dimensional.

1. We assume that the intersection of a planar domain Q by the x axis is a segment $[0, l]$. Here (x, y) are orthogonal Cartesian coordinates in the plane. We divide the segment $[0, l]$ into n equal parts and set $\varepsilon = l/n$. If n is large, ε is a small parameter. As a result, the periodic cell coincides with the segment $[0, 1]$. Using the Kirchhoff–Love hypotheses, we write the relations between the moments and curvatures for an orthotropic material in the form

$$M_{11} = -\Big(D_{11}\frac{\partial^2 w}{\partial x^2} + D_{12}\frac{\partial^2 w}{\partial y^2}\Big), \qquad M_{22} = -\Big(D_{12}\frac{\partial^2 w}{\partial x^2} + D_{22}\frac{\partial^2 w}{\partial y^2}\Big), \quad M_{12} = -2D_{66}\frac{\partial^2 w}{\partial x \partial y}.$$

Here $w(x, y)$ is the deflection (the assumption that the material is orthotropic is used only to simplify the formulas; subsequent calculations can be performed without this assumption), $D_{ij} = D_{ij}(x/\varepsilon)$ $(i, j = 1, 2, 6)$ are functions of the fast variable $\eta = x/\varepsilon$ that have a period equal to unity. Moreover, we assume that there is a positive constant γ such that the following inequalities (the condition of a positive definite stiffness matrix)

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hold:

$$D_{11}D_{22} - D_{12}^2 > \gamma > 0, \quad D_{66} > \gamma > 0.$$

The deflection is determined from the equation

$$-\frac{\partial^2}{\partial x^2}\Big(D_{11}\frac{\partial^2 w}{\partial x^2} + D_{12}\frac{\partial^2 w}{\partial y^2}\Big) - \frac{\partial^2}{\partial y^2}\Big(D_{12}\frac{\partial^2 w}{\partial x^2} + D_{22}\frac{\partial^2 w}{\partial y^2}\Big) - 4\frac{\partial^2}{\partial x \partial y}\Big(D_{66}\frac{\partial^2 w}{\partial x \partial y}\Big) = f. \tag{1.1}$$

We specify the following conjugation conditions for $\eta = 1/2$:

$$-M_{11}(x, y, 1/2 + 0) = \frac{\alpha}{\varepsilon}\Big[\frac{\partial w}{\partial x}\Big], \qquad -M_{11}(x, y, 1/2 - 0) = \frac{\alpha}{\varepsilon}\Big[\frac{\partial w}{\partial x}\Big]; \tag{1.2}$$

$$[w] = 0, \qquad [N_{11}] = 0. \tag{1.3}$$

Conditions (1.3) imply that the deflection and transverse shear force $N_{11} = -\partial M_{11}/\partial x - \partial M_{12}/\partial y$ are continuous across the boundary. We assume that the periodic cell of the straight line $\eta = 1/2$ is divided into two layers, whose mechanical characteristics can be the same. The coefficient α is positive and is called the stiffness of the hinge [1]. Henceforth, square brackets in formulas designate a jump of the function for $\eta = 1/2$. We seek an asymptote of the solution in the form

$$w^\varepsilon = \sum_{n=0}^{\infty} \varepsilon^n w^n(x, y, \eta). \tag{1.4}$$

The partial derivative $\partial/\partial x$ in (1.1) is replaced by the total derivative

$$\frac{d}{dx} = \frac{\partial}{\partial x} + \frac{1}{\varepsilon}\frac{\partial}{\partial \eta},$$

since we differentiate a composite function. Moreover, the second derivative with respect to x is written as

$$\frac{d^2}{dx^2} = \frac{\partial^2}{\partial x^2} + \frac{2}{\varepsilon}\frac{\partial^2}{\partial x \partial \eta} + \frac{1}{\varepsilon^2}\frac{\partial^2}{\partial \eta^2},$$

and Eq. (1.1) takes the form

$$\Big(\frac{\partial^2}{\partial x^2} + \frac{2}{\varepsilon}\frac{\partial^2}{\partial x \partial \eta} + \frac{1}{\varepsilon^2}\frac{\partial^2}{\partial \eta^2}\Big)\Big(D_{11}\Big(\frac{\partial^2}{\partial x^2} + \frac{2}{\varepsilon}\frac{\partial^2}{\partial x \partial \eta} + \frac{1}{\varepsilon^2}\frac{\partial^2}{\partial \eta^2}\Big)w^\varepsilon + D_{12}\frac{\partial^2 w^\varepsilon}{\partial y^2}\Big)$$
$$+4\Big(\frac{\partial}{\partial x} + \frac{1}{\varepsilon}\frac{\partial}{\partial \eta}\Big)\frac{\partial}{\partial y}\Big(D_{66}\Big(\frac{\partial}{\partial x} + \frac{1}{\varepsilon}\frac{\partial}{\partial \eta}\Big)\frac{\partial w^\varepsilon}{\partial x_2}\Big)$$
$$+\frac{\partial^2}{\partial y^2}\Big(D_{12}\Big(\frac{\partial^2}{\partial x^2} + \frac{2}{\varepsilon}\frac{\partial^2}{\partial x \partial \eta} + \frac{1}{\varepsilon^2}\frac{\partial^2}{\partial \eta^2}\Big)w^\varepsilon + D_{22}\frac{\partial^2 w^\varepsilon}{\partial y^2}\Big) = -f. \tag{1.5}$$

Substitution of (1.4) into (1.5) yields a recursively coupled system of equations for functions $w^n(x, y, \eta)$. To construct an averaged equation, it is necessary to determine only a few leading terms of the expansion. Setting to zero the coefficient of ε^{-4}, we obtain the equation

$$\frac{\partial^2}{\partial \eta^2}\Big(D_{11}\frac{\partial^2 w^0}{\partial \eta^2}\Big) = 0,$$

which implies that $w^0(x, y, \eta) = w^0(x, y)$. The function $w^1(x, y, \eta)$ is determined from the condition that the coefficient of ε^{-3} vanish:

$$\frac{\partial^2}{\partial \eta^2}\Big(D_{11}\frac{\partial^2 w^1}{\partial \eta^2}\Big) + \frac{\partial^2}{\partial \eta^2}\Big(2D_{11}\frac{\partial^2 w^0}{\partial x \partial \eta}\Big) + 2\frac{\partial^2}{\partial x \partial y}\Big(D_{11}\frac{\partial^2 w^0}{\partial y^2}\Big) = 0.$$

Hence, it follows that w^1 is independent of η. Similarly, the equation for the function $w^2(x, y, \eta)$ can be obtained from the requirement that the coefficient of ε^{-2} vanish:

$$\frac{\partial^2}{\partial \eta^2}\Big(D_{11}\frac{\partial^2 w^2}{\partial \eta^2}\Big) + \frac{\partial^2}{\partial \eta^2}\Big(2D_{11}\frac{\partial^2 w^1}{\partial x \partial \eta}\Big) + 2\frac{\partial^2}{\partial x \partial y}\Big(D_{11}\frac{\partial^2 w^1}{\partial y^2}\Big) + \frac{\partial^2}{\partial \eta^2}\Big(D_{11}\frac{\partial^2 w^0}{\partial x^2} + D_{12}\frac{\partial^2 w^0}{\partial \eta^2}\Big)$$

$$+\frac{\partial^2}{\partial x^2}\Big(D_{11}\frac{\partial^2 w^0}{\partial \eta^2}\Big)+4\frac{\partial^2}{\partial \eta\partial y}\Big(D_{66}\frac{\partial^2 w^0}{\partial \eta\partial y}\Big)+\frac{\partial^2}{\partial y^2}\Big(D_{12}\frac{\partial^2 w^0}{\partial \eta^2}\Big)=0.$$

Therefore, the function $w^2(x,y,\eta)$ satisfies the relation

$$\frac{\partial^2}{\partial \eta^2}\Big(D_{11}\frac{\partial^2 w^2}{\partial \eta^2}+D_{11}\frac{\partial^2 w^0}{\partial x^2}+D_{12}\frac{\partial^2 w^0}{\partial y^2}\Big)=0. \tag{1.6}$$

We now substitute the function w^ε from formula (1.4) into the conjugation condition (1.2). Introducing the fast variable η, one obtains this condition in the form

$$D_{11}\Big(\frac{\partial^2}{\partial x^2}+\frac{2}{\varepsilon}\frac{\partial^2}{\partial x\partial \eta}+\frac{1}{\varepsilon^2}\frac{\partial^2}{\partial \eta^2}\Big)+D_{12}\frac{\partial^2 w^\varepsilon}{\partial y^2}\Big|_{\eta=1/2}=\frac{\alpha}{\varepsilon}\Big[\Big(\frac{\partial}{\partial x}+\frac{1}{\varepsilon}\frac{\partial}{\partial \eta}\Big)w^\varepsilon\Big].$$

Equating coefficients with the same powers of ε on the left and right sides of the last equality, we obtain the following relations for jumps in the functions w^0, w^1, and w^2:

$$D_{11}\frac{\partial^2 w^0}{\partial \eta^2}\Big|_{\eta=1/2}=\alpha[w^0],\qquad D_{11}\frac{\partial^2 w^1}{\partial \eta^2}+2D_{11}\frac{\partial^2 w^0}{\partial x\partial \eta}\Big|_{\eta=1/2}=\alpha\Big[\frac{\partial w^0}{\partial x}+\frac{\partial w^1}{\partial \eta}\Big],$$
$$\Big[D_{11}\frac{\partial^2 w^2}{\partial \eta^2}+D_{11}\frac{\partial^2 w^0}{\partial x^2}+D_{12}\frac{\partial^2 w^0}{\partial y^2}\Big]\Big|_{\eta=1/2}=\alpha\Big[\frac{\partial w^2}{\partial \eta}+\frac{\partial w^1}{\partial x}\Big]. \tag{1.7}$$

It follows from Eq. (1.6) and the conjugation condition (1.7) that

$$D_{11}\frac{\partial^2 w^2}{\partial \eta^2}+D_{11}\frac{\partial^2 w^0}{\partial x^2}+D_{12}\frac{\partial^2 w^0}{\partial y^2}=\eta\varphi_1(x,y)+\varphi_2(x,y),$$

where $\varphi_1(x)$ and $\varphi_2(x)$ are unknown functions.

The continuity of the deflection and transverse shear force at the interface implies that $\varphi_1(x,y)$ can be set equal to zero and the function $\varphi_2(x,y)$ has the form

$$\varphi_2(x,y)=\frac{\alpha}{1+\alpha\lambda_{11}}\Big(\frac{\partial^2 w^0}{\partial x^2}+\lambda_{12}\frac{\partial^2 w^0}{\partial y^2}\Big).$$

Here

$$\lambda_{11}=\int_0^1\frac{1}{D_{11}(s)}\,ds,\quad \lambda_{12}=\int_0^1\frac{D_{12}(s)}{D_{11}(s)}\,ds,\quad \mu=\int_0^1\Big(D_{22}(s)-\frac{D_{12}(s)^2}{D_{11}(s)}\Big)\,ds,\quad \lambda_{66}=\int_0^1 D_{66}(s)\,ds.$$

Setting the coefficient of ε^0 in Eq. (1.5) equal to zero, we obtain the following averaged relations between the moments and curvatures:

$$M_{11}^0=-\Big(\frac{\alpha}{1+\alpha\lambda_{11}}\frac{\partial^2 w^0}{\partial x^2}+\frac{\alpha\lambda_{12}}{1+\alpha\lambda_{11}}\frac{\partial^2 w^0}{\partial y^2}\Big),$$
$$M_{22}^0=-\Big[\frac{\alpha\lambda_{12}}{1+\alpha\lambda_{11}}\frac{\partial^2 w^0}{\partial x^2}+\Big(\mu+\frac{\alpha\lambda_{12}^2}{1+\alpha\lambda_{11}}\Big)\frac{\partial^2 w^0}{\partial y^2}\Big],$$
$$M_{12}^0=-2\lambda_{66}\frac{\partial^2 w^0}{\partial x\partial y},$$

and the averaged equation

$$\frac{\alpha}{1+\alpha\lambda_{11}}\Big(\frac{\partial^2}{\partial x^2}+\lambda_{12}\frac{\partial^2}{\partial y^2}\Big)\Big(\frac{\partial^2 w^0}{\partial x^2}+\lambda_{12}\frac{\partial^2 w^0}{\partial y^2}\Big)+\mu\frac{\partial^4 w^0}{\partial y^4}+4\lambda_{66}\frac{\partial^4 w^0}{\partial x^2\partial y^2}=-f. \tag{1.8}$$

2. Equation (1.8) retains the elliptic property of the initial equation (1.1). For small α, it is singularly perturbed. Considering the limiting case of Eq. (1.8) for $\alpha\to +0$ and denoting the limit by u, we obtain the equation

$$\mu \frac{\partial^4 u}{\partial y^4} + 4\lambda_{66} \frac{\partial^4 u}{\partial x^2 \partial y^2} = -f. \tag{2.1}$$

Equation (2.1) is of a composite type with a family of real characteristics $x = \text{const}$ of multiplicity 2 and a pair of complex conjugate characteristics. Similar equations were studied, for example, in [4]. We set $w^0 = u^\alpha$ in Eq. (1.8) and formulate the first boundary-value problem

$$u^\alpha \Big|_{\partial Q} = 0, \qquad \frac{\partial u^\alpha}{\partial n}\Big|_{\partial Q} = 0. \tag{2.2}$$

One can readily prove that a solution of the original problem (1.8) and (2.2) converges to a solution of the limiting problem. We now consider exact formulations. Let Q be a bounded planar domain with a piecewise-smooth boundary ∂Q. The following symmetric bilinear form is naturally associated with the boundary-value problem (1.8), (2.2):

$$a_\alpha(u^\alpha, v) = \int\limits_Q \frac{\alpha}{1+\alpha\lambda_{11}} \left(\frac{\partial^2 u^\alpha}{\partial x^2} + \lambda_{12}\frac{\partial^2 u^\alpha}{\partial y^2}\right)\left(\frac{\partial^2 v}{\partial x^2} + \lambda_{12}\frac{\partial^2 v}{\partial y^2}\right) dx\, dy$$

$$+ \int\limits_Q \left(\mu \frac{\partial^2 u^\alpha}{\partial y^2}\frac{\partial^2 v}{\partial y^2} + 4\lambda_{66}\frac{\partial^2 u^\alpha}{\partial x \partial y}\frac{\partial^2 v}{\partial x \partial y}\right) dx\, dy. \tag{2.3}$$

Obviously, expression (2.3) can be written as

$$a_\alpha(u^\alpha, v) = \frac{\alpha}{1+\alpha\lambda_{11}} a_1(u^\alpha, v) + a_0(u^\alpha, v),$$

where

$$a_1(u^\alpha, v) = \int\limits_Q \left(\frac{\partial^2 u^\alpha}{\partial x^2} + \lambda_{12}\frac{\partial^2 u^\alpha}{\partial y^2}\right)\left(\frac{\partial^2 v}{\partial x^2} + \lambda_{12}\frac{\partial^2 v}{\partial y^2}\right) dx\, dy,$$

$$a_0(u^\alpha, v) = \int\limits_Q \left(\mu \frac{\partial^2 u^\alpha}{\partial y^2}\frac{\partial^2 v}{\partial y^2} + 4\lambda_{66}\frac{\partial^2 u^\alpha}{\partial x \partial y}\frac{\partial^2 v}{\partial x \partial y}\right) dx\, dy.$$

Let $f \in L^2(Q)$. The boundary-value problem (1.8), (2.2) admits the variational formulation: Find a function $u^\alpha \in H_0^2(Q)$ such that

$$a_\alpha(u^\alpha, v) = (f, v) \tag{2.4}$$

for any $v \in H_0^2(Q)$. Here (f, v) is the scalar product in $L^2(Q)$. It is known [5] that the function u^α is determined from (2.4) uniquely. For $\alpha = +0$, we obtain the bilinear symmetric form $a_0(u^0, v)$ and the corresponding quadratic form $a_0(u^0, u^0)$. We introduce a Hilbert space V by supplementing the set of functions of class $C_0^\infty(Q)$ by the norm

$$\|u\|_V = \left[\int\limits_Q \left(\left(\frac{\partial^2 u}{\partial y^2}\right)^2 + \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + u^2\right) dx\, dy\right]^{1/2}. \tag{2.5}$$

A function $u^0 \in V$ that satisfies the integral identity

$$a_0(u^0, v) = (f, v) \tag{2.6}$$

for any $v \in V$ and $f \in L^2(Q)$ is called a weak solution of the boundary-value problem for Eq. (2.1). For functions from V, the Poincaré inequality [4] holds:

$$\int\limits_Q v^2\, dx\, dy \leqslant C \int\limits_Q \left(\frac{\partial v}{\partial x}\right)^2 dx\, dy.$$

A similar inequality is valid for the derivative with respect to y. (Here and below, the constants C with or without subscripts are independent of the functions.) Consequently, for $v \in C_0^\infty(Q)$, the semi-norm

$$|v|_V^2 = \int\limits_Q \left[\left(\frac{\partial^2 v}{\partial y^2}\right)^2 + \left(\frac{\partial^2 v}{\partial x \partial y}\right)^2\right] dx\, dy$$

is equivalent to norm (2.5). Furthermore, a unique function u^0 that is a weak solution of problem (2.6) exists. Indeed, for fixed $f \in L^2(Q)$, the bilinear form (u^0, f) defines a linear continuous functional on V. At the same time, the inequality $a_0(u^0, u^0) \geqslant C_0|u^0|_V^2 \geqslant C_1\|u^0\|_V^2$ holds. Here the positive constants C_0 and C_1 are independent of u^0. Therefore, according to the Lax-Milgram lemma, there is a unique element $u^0 \in V$ such that $a_0(u^0, v) = (f, v)$ for any function $v \in V$, and this is the desired solution. In addition, $u^0(x, y)$ satisfies the boundary conditions

$$u^0\Big|_{\partial Q} = 0, \quad \frac{\partial u^0}{\partial y}\Big|_{\partial Q^*} = 0.$$

Here ∂Q^* is the noncharacteristic part of the boundary. We now examine the convergence of u^α to u^0.

Theorem. *As $\alpha \to +0$, a weak solution of problem* (1.8) *and* (2.1) *converges to a weak solution of problem* (2.6).

Indeed, we have the chain of inequalities

$$a_\alpha(u^\alpha, u^\alpha) = a_0(u^\alpha, u^\alpha) + \frac{\alpha}{1+\alpha\lambda_{11}} a_1(u^\alpha, u^\alpha) = (f, u^\alpha) \leqslant \|f\|_0 \|u^\alpha\|_0 \leqslant \|f\|_0 \|u^\alpha\|_V. \tag{2.7}$$

It follows from (2.7) that

$$\|u^\alpha\|_V \leqslant C_2, \qquad \alpha\left\|\left(\frac{\partial^2 u^\alpha}{\partial x^2} + \lambda_{12}\frac{\partial^2 u^\alpha}{\partial y^2}\right)\right\|_0^2 \leqslant C_3. \tag{2.8}$$

Here $\|u\|_0$ denotes the norm in $L^2(Q)$. Estimate (2.8) implies that the sequence u^α belongs to a bounded subset of the space V. Hence, from the sequence u^α we can separate a slowly converging subsequence in V, for which we retain the adopted notation. It follows from the second of estimates (2.8) that for any $v \in C_0^\infty(Q)$, the term with α in the integral identity (2.1) converges to zero. Indeed, for an arbitrary function from $C_0^\infty(Q)$, we have

$$\frac{\alpha}{1+\alpha\lambda_{11}}\left|\int\limits_Q \left(\frac{\partial^2 u^\alpha}{\partial x^2} + \lambda_{12}\frac{\partial^2 u^\alpha}{\partial y^2}\right)\left(\frac{\partial^2 v}{\partial x^2} + \lambda_{12}\frac{\partial^2 v}{\partial y^2}\right) dx\, dy\right|$$

$$\leqslant \alpha C\left\|\frac{\partial^2 u^\alpha}{\partial x^2} + \lambda_{12}\frac{\partial^2 u^\alpha}{\partial y^2}\right\|_0 \left\|\frac{\partial^2 v}{\partial x^2} + \lambda_{12}\frac{\partial^2 v}{\partial y^2}\right\|_0 \leqslant \frac{\sqrt{\alpha}}{\sqrt{1+\alpha\lambda_{11}}} C_3 \left\|\frac{\partial^2 v}{\partial x^2} + \lambda_{12}\frac{\partial^2 v}{\partial y^2}\right\|_0.$$

Here the constants C and C_3 are independent of the functions v and u^α. The last inequality makes it possible to pass to the limit for the chosen subsequence in the integral identity (2.4). Thus, the limiting function satisfies the integral identity (2.6). Since the liming problem has a unique solution, the subsequence has the same limit.

3. A similar convergence theorem can also be proved for a mixed problem where the deflection and slope are specified on one part of the boundary and the moments and transverse shear forces on the other part. However, if the natural boundary conditions are formulated for the limiting problem, a similar theorem cannot be proved, since in contrast to the elliptic case, the space of solutions of the homogeneous problem is infinite-dimensional. Inasmuch as the coefficients in Eq. (2.1) are constant, they can be set equal to unity by appropriate extension of the coordinates. We set $w^0 = u$ and write Eq. (2.1) in the form

$$\frac{\partial^4 u}{\partial y^4} + \frac{\partial^4 u}{\partial x^2 \partial y^2} = f. \tag{3.1}$$

Let $f = 0$. The general solution of the homogeneous equation (3.1) has the form

$$u(x, y) = y\varphi_1(x) + \varphi_2(x) + W(x, y),$$

where $W(x,y)$ is a harmonic function and $\varphi_1(x)$ and $\varphi_2(x)$ are arbitrary functions. We first consider a particular case where the domain is the upper half-plane: $Q = \{(x,y), y \geqslant 0\}$. The natural boundary conditions have the form

$$\frac{\partial \Delta u}{\partial y}(x,+0) = f_1(x), \qquad \frac{\partial^2 u}{\partial y^2}(x,+0) = f_2(x). \tag{3.2}$$

Here Δu is the Laplacian of the function u. We assume that f_1 and f_2 have a compact support. Then,

$$\varphi_1(x) = \frac{1}{2} \int\limits_{-\infty}^{+\infty} f_1(t)|x-t|\, dt,$$

and $W(x,y)$ is represented by the convolution:

$$W(x,y) = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} K(x-t,y) f_2(t)\, dt.$$

Here the kernel $K(x-t,y)$ is calculated from the formula

$$K(x-t,y) = y \ln[(x-t)^2 + y^2] + 2y - 2(x-t) \arctan \frac{y}{x-t}.$$

Moreover, $W(x,y)$ satisfies the boundary condition

$$\frac{\partial^2 W}{\partial y^2}(x,+0) = f_2(x)$$

and, hence, it is determined with accuracy up to the term $ax+b$, where a and b are arbitrary constants. The function $\varphi_2(x)$ cannot be determined from boundary condition (3.2); consequently, the solution space of the homogeneous problem is infinite-dimensional. If the domain Q is arbitrary, the natural boundary conditions have the form

$$\left.\frac{\partial}{\partial n} \frac{\partial u}{\partial y}\right|_{\partial Q} = f_1(s), \qquad \left.\frac{\partial}{\partial y} \Delta u \cos(n,y)\right|_{\partial Q} = f_2(s).$$

Here $f_1(s)$ and $f_2(s)$ are specified functions of the arc length. The general solution of the homogeneous equation with homogeneous natural boundary conditions has the form $u = d_1 x + d_2 y + \varphi(x)$ and, therefore, it is determined with accuracy up to the arbitrary function $\varphi(x)$.

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